CHAPTER : 7

KINEMATICS

Dynamics # It is that branch of mechanics which deals with the motion of objects

Dynamics is divided into branches

Dynamics

Kinematics   Kinetics:

Kinenamatics # That branch of dynamics which deals only with the geometry or nature of motion of a body but not with the effect of forces on the body.

Kinetics # that branch of dynamics which deals with the motion as well as the effect of forces on the motion i.e it establishes relations between the motion of the body and the forces under which the motion takes place.

Particle # A particle is a point body i.e a body which has no dimensions.

Note # This definition of particle is an ideal definition because no such body is found in nature. But a body of finite dimensions can be considered as a particle if motion of body is described in terms

quantities which are quite larger as compared to the dimensions of body. e.g. In describing the motion of the moon with respect to earth, the moon can be treated as a particle because the distance of the moon from the earth is about 2,40,000 miles which is very large as compared to the radius of the motion.

Path of the Particle

The position of particle is represented by a vector $\underline{r}$ whose initial point is at the origin and the terminal point is at particle. If the particle is moving, the vector changes with time i.e it is a function of time.

The curve that is traced by the particle with the passage of time is called trajectory or the path or the orbit of the particle.

Path of the particle can be given by

$$\underline{r} = \underline{r}(t)$$

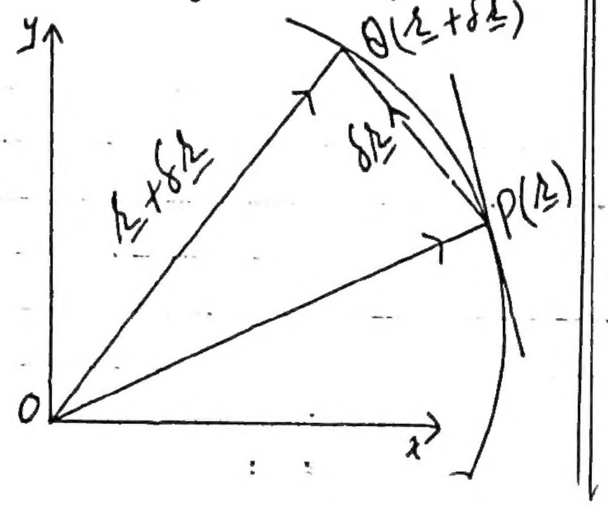
or by three scalar quantities

$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

Velocity and Acceleration

Velocity of a particle is the rate of change of displacement w.r.t time.

Suppose a particle moves along a plane curve and $P(\underline{r})$, $Q(\underline{r}+\delta\underline{r})$ be its position at times t, $t+\delta t$ i.e



At time t displacement of particle from O is $\overrightarrow{OP} = \vec{r}$ and at time $t+\delta t$ its displacement from O is $\overrightarrow{OQ} = \vec{r}+\delta \vec{r}$

$$\text{Change in displacement} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = \delta \underline{r} = \underline{r}+\delta \underline{r} - \underline{r}$$

$\overrightarrow{PQ} = \delta \underline{r}$ which is displacement of particle from point P in small time interval δt.

Average rate of change of displacement

= Average velocity of particle during time interval δt $= \dfrac{\delta \underline{r}}{\delta t} = \dfrac{\overrightarrow{PQ}}{\delta t}$

As $\delta t \to 0$, the direction of $\delta \underline{r} = \overrightarrow{PQ}$ approaches the direction of tangent to curve at P and $\dfrac{PQ}{\delta t} = \dfrac{|\delta \underline{r}|}{\delta t}$ approaches the speed of the particle at P.

Thus $\lim\limits_{\delta t \to 0}\left(\dfrac{\delta \underline{r}}{\delta t}\right)$ is a vector whose magnitude is the speed and whose direction is the direction of motion of a particle whose position vector is $\underline{r}$ at time t i.e

$\dfrac{d\underline{r}}{dt} = \vec{v}$, the velocity vector of the particle.

<u>Magnitude</u> If $\overset{\frown}{PQ} = \delta s$, then magnitude of the velocity or speed of particle is given by

$$|\vec{v}| = \left|\frac{d\underline{r}}{dt}\right| = \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{|\delta \underline{r}|}{\delta t} = \lim_{\delta t \to 0} \frac{\delta s}{dt} = \frac{ds}{dt}$$

Since when $\delta t \to 0$, $Q \to P$, then $|\delta \underline{r}| = |\overrightarrow{PQ}| = \overset{\frown}{PQ} = \delta s$

Acceleration

Acceleration of a particle is the rate of change of velocity w.r.t time.

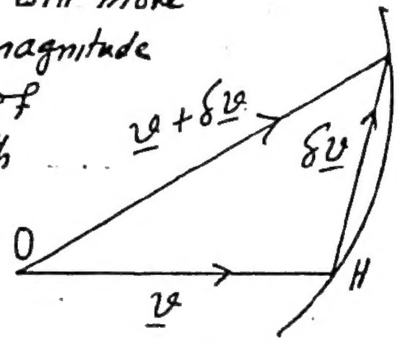
If $\underline{v}$, $\underline{v}+\delta\underline{v}$ are velocities of a particle at P and Q respectively, then acceleration vector $\underline{a}$ is defined as

$$\underline{a} = \lim_{\delta t\to 0}\frac{\delta \underline{v}}{\delta t} = \frac{d\underline{v}}{dt}$$

$$\underline{a} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2\underline{r}}{dt^2}$$

Hodograph

Let O be a fixed point. Let vector $\overrightarrow{OH}$ represents the velocity of the particle at any time t. The point H will move continuously with time if the magnitude of the velocity or the direction of the velocity or both change with time. The curve traced by H is called the hodograph of the motion of the particle. The accelration at any time t when velocity of particle is given by



$$\underline{a} = \lim_{\delta t\to 0}\frac{\delta \underline{v}}{\delta t} = \frac{d\underline{v}}{dt}$$

and is along the tangent to the hodograph. Thus we can define the hodograph of moving particle as

If the velocity vectors of a moving particle are laid off from a fixed point, the extremeties of these vectors trace out a curve which is called the hodograph of moving particle.

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Note # 1) Direction of acceleration is the direction in which the velocity changes.

2) # The velocity vector is defined in terms of magnitude and direction only i.e velocity vector is a free vector

3) # An acceleration vector is free vector but displacement from a specified point is a fixed vector.

Cartesian Components of Velocity and Acceleration

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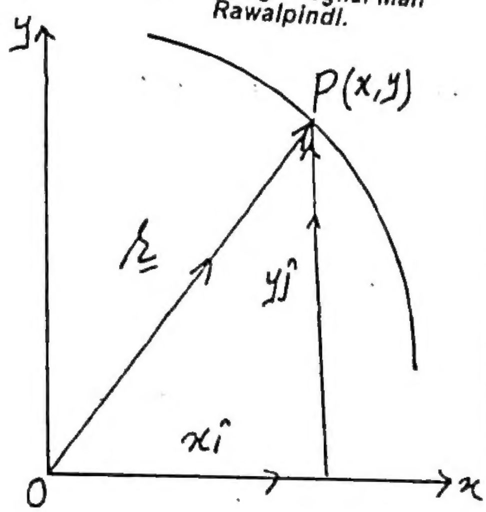
Let $P(x,y)$ be the position of particle at time t moving along a curve in the xy-plane then



$$\underline{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

Here $v_x = \frac{dx}{dt} = \dot{x}$ & $v_y = \frac{dy}{dt} = \dot{y}$

are called cartesian components of velocity along axes

The magnitude of velocity is given by

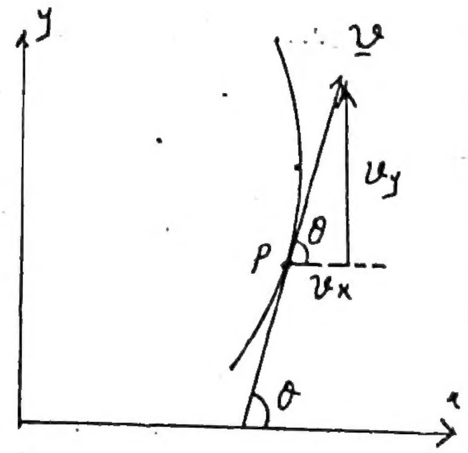
$$v = |\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

$\because (ds)^2 = (dx)^2 + (dy)^2$

where s is the distance of the particle along the path from some fixed point on the path.

Direction of velocity

If θ is the angle of velocity vector with x-axis, then



$$\tan\theta = \frac{v_y}{v_x} = \frac{dy/dt}{dx/dt}$$

$$= dy/dx = \text{slope of tangent}$$

at point. Therefore the direction of $\underline{v}$

is the direction of the tangent at P to the path of the particle. This is the direction of motion of the particle

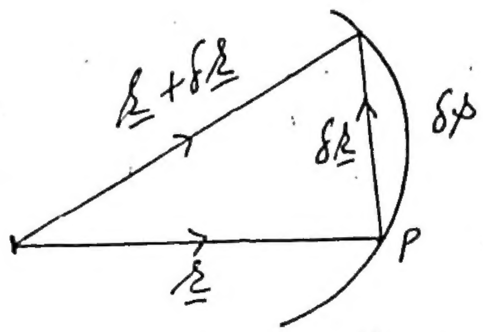
Also

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds} \cdot \frac{ds}{dt}$$

$$= v \frac{d\underline{r}}{ds}$$

Since $\frac{d\underline{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta \underline{r}}{\delta s}$

Therefore this vector is parallel to the tangent at P and its magnitude is



$$\left| \lim_{\delta s \to 0} \frac{\delta \underline{r}}{\delta s} \right|$$

$$= \lim_{\delta s \to 0} \frac{|\delta \underline{r}|}{\delta s} = 1$$

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Since when $Q \to P$, $|\delta \underline{r}| = \delta s$

Thus this is a unit vector along tangent and is denoted by $\hat{t}$ i.e

$$\frac{d\underline{r}}{ds} = \frac{d\underline{r}}{ds} = \hat{t} \dots$$

$$\underline{v} = v\hat{t}$$

This equation shows that at any instant the particle is moving in the direction of the tangent to the path.

Cartesian Components of Acc

$$\underline{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)$$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

Here $a_x = \frac{d^2x}{dt^2}$, $a_y = \frac{d^2y}{dt^2}$ are called cartesian components of acceleration

Magnitude of acceleration $= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$

If the direction of acceleration makes an angle ϕ with x-axis, then

$$\tan\phi = \frac{dy}{dx} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

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Angular Motion of Rigid Body

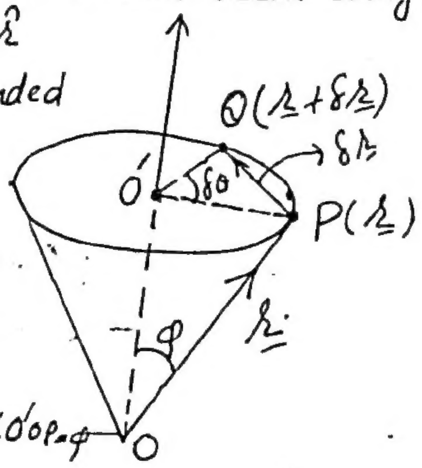
When a rigid moves on straight, then all of its particles cover same displacement during any interval of time.

Suppose a rigid body rotates about an axis through point O of body. Each point (particle) on the axis of rotation remains fixed on it and the other points move with different speeds. The points of body farther from the axis of rotation move faster than the points of body nearer to the axis of rotation.

Angular Velocity Vector

Let $P(\underline{r})$ and and $Q(\underline{r}+\delta\underline{r})$ be the positions at time $t, t+\delta t$ respectively of a particle of body which is rotating about an axis through point O. Let $\hat{a}$ be unit vector along the axis of rotation. Then $\hat{a}, \hat{r}$ and $\hat{a}\times\hat{r}$ form a right handed system. Clearly particle moves along a circle with centre O′ on the axis of rotation. The radius of this circular path is

$O'P = r\sin\phi$ where $\angle O'OP = \phi$



Let $\delta\theta = \angle PO'Q$ be the infinitesimal angular displacement during time δt, then the angular velocity of the particle is defined as

$$\omega = \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt}$$

The vector $\underline{\omega} = \frac{d\theta}{dt}\hat{a}$ is called the vector angular velocity.

Relation between Linear & Vector Angular Velocity

The linear velocity v of particle at P is along the tangent at P and $\hat{a} \times \hat{r}$ is a unit tangent vector at P.

Now $|\delta \underline{r}| = \widehat{PQ} = r\sin\varphi\,\delta\theta$

$$v = \lim_{\delta t \to 0} \frac{|\delta \underline{r}|}{\delta t}$$

$$= \lim_{\delta\theta \to 0} \left(r\sin\varphi \frac{\delta\theta}{\delta t}\right)$$

$$= r\sin\varphi . \omega$$

$$\underline{v} = v\,\hat{a}\times\hat{r} \quad \text{(vector = magnitude} \times \text{unit vector)}$$

$$= r\sin\varphi\,\omega\,\hat{a}\times\hat{r}$$

$$= \sin\varphi\,\omega\,\hat{a}\times r\hat{r}$$

$$= (r\sin\varphi\,\omega)\,\hat{a}\times\hat{r}$$

$$\underline{v} = \underline{\omega}\times\underline{r} \longrightarrow (1)$$

Deduction # As particle describes circular motion, therefore OP serves as the generator

of the Cone as shown in fig where

$$|\overrightarrow{OP}| = r$$

remains Constant. From equation ①, we have.

$$\underline{v} = \underline{\omega} \times \underline{r}$$

$$\Rightarrow \frac{d\underline{r}}{dt} = \underline{\omega} \times \underline{r}$$

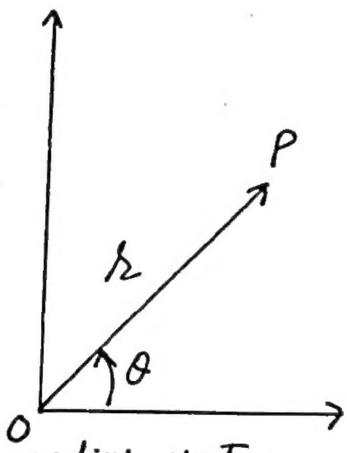
$$\Rightarrow \frac{d}{dt}(r\hat{r}) = \underline{\omega} \times r\hat{r} \qquad \text{as } \underline{r} = r\hat{r}$$

$$\Rightarrow r\frac{d\hat{r}}{dt} = r(\underline{\omega} \times \hat{r}) \qquad \because r \text{ is Constant}$$

$$\Rightarrow \frac{d\hat{r}}{dt} = \underline{\omega} \times \hat{r}$$

Plane Polar Co-ordinates

If a particle is moving anti-clockwise and is at point P of its path. Let $\angle XOP = \theta$ and $|\overrightarrow{OP}| = r$, then (r, θ) are called polar co-ordinates of point P



Radial Direction

The direction of radius vector is called radial direction

Transverse Direction

The direction perpendicular to the radius vector and in the sense of increasing θ is called transverse direction

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Radial and Transvese components of Velocity and acceleration

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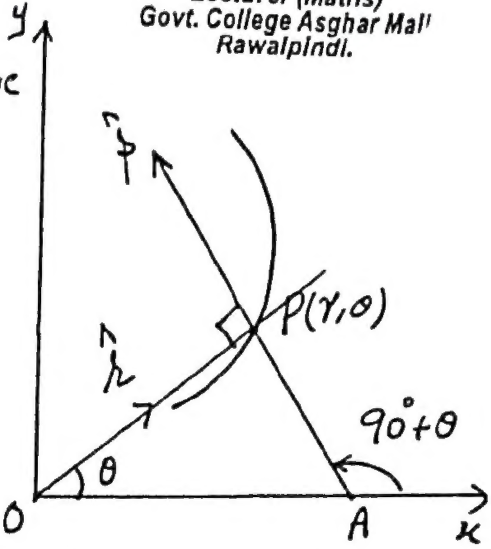
Let $P(r,\theta)$ be the position of particle moving in xy-plane and $\hat{r}, \hat{s}$ be be the unit vectors along radial and transverse directions at point P.



$\because$ External angle of the triangle is sum of the opposite internal angle

$\therefore \angle POX = 90^\circ + \theta$

$$\underline{r} = x\hat{i} + y\hat{j}$$

But $x = r\cos\theta$

$y = r\sin\theta$

$$\Rightarrow \underline{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

$$\frac{\underline{r}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\Rightarrow \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \longrightarrow ①$$

Now angle of $\hat{s}$ with x-axis is $90^\circ+\theta$ as shown in fig. Therefore

$$\hat{s} = \cos(90^\circ+\theta)\,\hat{i} + \sin(90^\circ+\theta)\,\hat{j}$$

$$\hat{s} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \longrightarrow ②$$

Differenting ① & ② w.r.t time t

$$\frac{d\hat{r}}{dt} = -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}$$

$$= (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\frac{d\theta}{dt}$$

$$= \hat{s}\dot{\theta} \qquad \text{By ②} \longrightarrow ③$$

$$d\hat{s} = -\cos\theta\,i\frac{d\theta}{dt} - \sin\theta\frac{d\theta}{dt}\hat{j}$$

$$= -(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})\frac{d\theta}{dt}$$

$$\frac{d\hat{s}}{dt} = -\hat{r}\dot{\theta} \quad \text{by } ① \longrightarrow ④$$

Velocity#

$$\because \underline{r} = r\hat{r}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\hat{r})$$

$$\underline{v} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

putting value of $\frac{d\hat{r}}{dt}$ from ③

$$\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{s} \longrightarrow ⑤$$

$\Rightarrow$ Radial Component of velocity $= v_r = \dot{r} = \frac{dr}{dt}$

Transverse component of velocity $= v_\theta = r\dot{\theta} = r\frac{d\theta}{dt}$

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Acceleration#

$$\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{s}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{s})$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}\frac{d\hat{s}}{dt}$$

putting values of $\frac{d\hat{r}}{dt}$ and $\frac{d\hat{s}}{dt}$

$$\underline{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{s} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}(-\dot{\theta}\hat{r})$$

$$= [\ddot{r} - r\dot{\theta}^2]\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{s}$$

$\Rightarrow$ Radial Component of acc $= a_r = \ddot{r} - r\dot{\theta}^2$

Transverse component of acc $= a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$$a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

Remarks# note that $\hat{r}, \hat{s}, \hat{k}$ form an orthogonal right-handed system. For different position of particle during its motion, the axes of $\hat{r}, \hat{s}$ ratate about z-axis with angular velocity

$$\underline{\omega} = \frac{d\theta}{dt}\hat{k}$$

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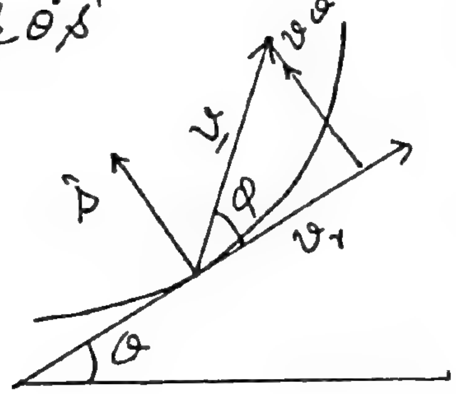
Magnitude and Direction of Velocity and Accleration in Polar Co-ordinates

Since $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{s}$

So $|\underline{v}| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$



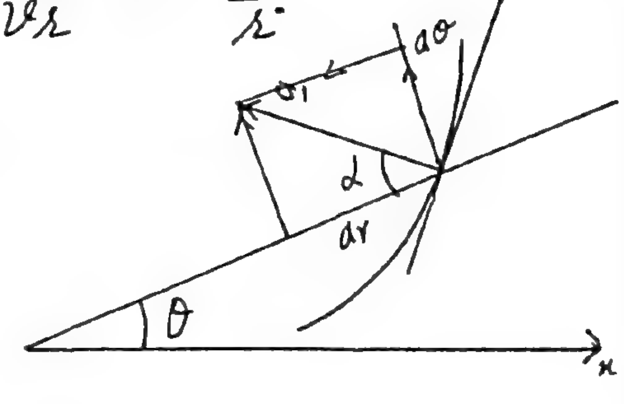
If φ is the angle betweeen the velocity vector $\underline{v}$ and the radial direction $\hat{r}$, then

$$\tan\varphi = \frac{v_\theta}{v_r} = \frac{r\dot{\theta}}{\dot{r}}$$

Also $\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{s}$

$= a_r\hat{r} + a_\theta\hat{s}$

$|\underline{a}| = \sqrt{(a_r)^2 + (a_\theta)^2}$



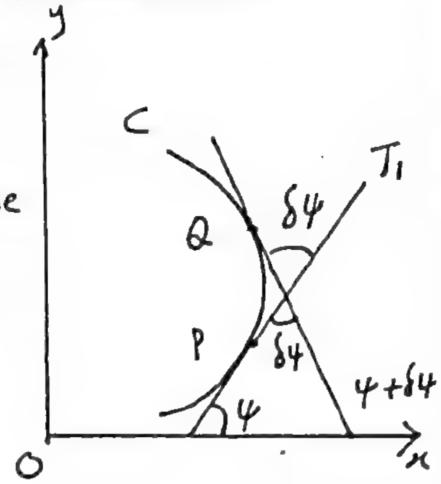
If α be the angle between $\underline{a}$ and radial

direction $\hat{r}$, then

$$\tan\alpha = \frac{d\theta}{dr} = \frac{r\ddot{\theta} + 2\dot{r}\dot{\theta}}{\ddot{r} - r\dot{\theta}^2}$$

Curvature and Radius of Curvature

Let $P(x,y)$ & $Q(x+\delta x, y+\delta y)$ be two neighbouring points on the plane curve C. Suppose tangent has rotated through an angle $\delta\psi$ in going from P to Q and $\overset{\frown}{PQ} = \delta s$. Then



(i) # $\delta\psi$ is total curvature of arc $\overset{\frown}{PQ}$

(ii) # $\dfrac{\delta\psi}{\delta s}$ is average curvature of arc $\overset{\frown}{PQ}$

(iii) # $\displaystyle\lim_{\substack{\delta s \to 0 \\ Q \to P}} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$ is called curvature

of the curve at point P and is denoted by K (kappa). Since curvature is taken in the direction of increasing angle ψ, therefore it +ve and we write

$$K = \left|\frac{d\psi}{ds}\right|$$

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The reciprocal of curvature at point P is called radius of curvature at point P and is denoted by ρ (rho). Thus

$$\rho = \frac{1}{K}$$

For cartesian curve $y = f(x)$ the formula

for curvature and radius of curvature is

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

$$\rho = \frac{1}{K} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

Intrinsic Co-ordinates

Let s be the arc length of a curve C from a fixed point P to some point Q(x,y) on the curve. Let ψ be the angle between the tangents at P and Q. Then s and ψ are called the intrinsic Co-ordinates of point Q(x,y)

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Note # Any relation between the intrinsic Co-ordinates s, ψ of the form

$$s = f(\psi)$$

is called intrinsic equation of the curve.

Tangential and Normal Directions

The direction of tangent at any point of the path is called tangential direction and the direction of perpendicular to the tangent and in the sense of increasing of inclination of tangent is called normal direction at that point.

Tangential and Normal Components of Velocity and Acceleration

Velocity

Let $P(\underline{r})$ and $Q(\underline{r}+\delta\underline{r})$ be the positions of particle moving in plane at time t and $t+\delta t$ respectively and

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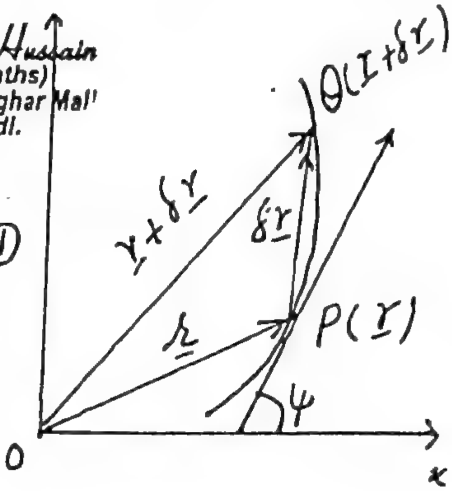
$$\widehat{PQ} = \delta s$$



$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d\underline{r}}{ds}\cdot\frac{ds}{dt} \longrightarrow (1)$$

(By chain rule)

Now

$$\frac{d\underline{r}}{ds} = \lim_{\delta s\to 0}\frac{\delta\underline{r}}{\delta s}$$

$$= \lim_{\delta s\to 0}\frac{1}{\delta s}(\delta\underline{r})$$ is along the tangent at point P. and

$$\left|\frac{d\underline{r}}{ds}\right| = \lim_{\delta s\to 0}\frac{|\delta\underline{r}|}{\delta s} = \lim_{\substack{\delta s\to 0\\ Q\to P}}\frac{|\overline{PQ}|}{\widehat{PQ}}$$

$$= \lim_{Q\to P}\frac{|\overline{PQ}|}{|\overline{PQ}|} = 1$$

{Since when P & Q are very close $|\overrightarrow{PQ}| = \widehat{PQ}$}

Thus $\frac{d\underline{r}}{ds}$ is a unit tangent vector at P.

Let $\frac{d\underline{r}}{ds} = \hat{t}$

Also let $\hat{n}$ unit vector along normal direction

at point P

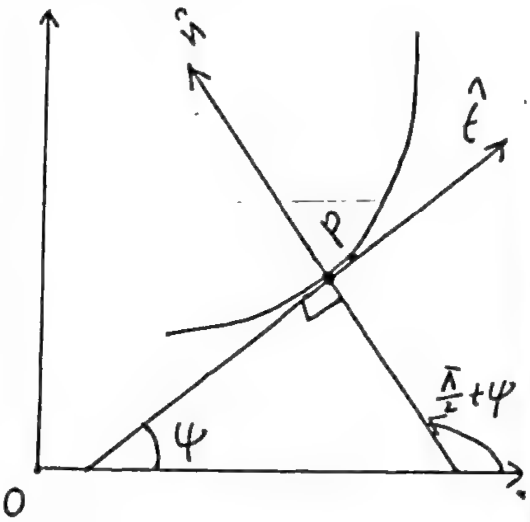
Then from ①

$$\underline{v} = \frac{ds}{dt} \cdot \frac{d\underline{r}}{ds} = v\hat{t} = v\hat{t} + 0\hat{n}$$

⇒ tangential Component of velocity $= v_t = v$

Normal Componet of velocity $= v_n = 0$

Thus we note that velocity of the particle is entirely along the tangent to the path.



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Acceleration

In Components form

$$\hat{t} = \cos\psi\,\hat{i} + \sin\psi\,\hat{j} \longrightarrow ②$$

and

$$\hat{n} = \cos(90^\circ+\psi)\hat{i} + \sin(90^\circ+\psi)\hat{j}$$

$$= -\sin\psi\,\hat{i} + \cos\psi\,\hat{j} \longrightarrow ③$$

$$\underline{v} = v\hat{t}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(v\hat{t})$$

$$\underline{a} = \frac{dv}{dt}\hat{t} + v\frac{d\hat{t}}{dt} \longrightarrow ③$$

from ② by differentiating w.r.t t

$$\frac{d\hat{t}}{dt} = -\sin\psi\,\hat{i}\cdot\frac{d\psi}{dt} + \cos\psi\,\hat{j}\,\frac{d\psi}{dt}$$

$$\frac{d\hat{t}}{dt} = (-\sin\psi\,\hat{i} + \cos\psi\,\hat{j})\frac{d\psi}{dt}$$

$$= \hat{n}\frac{d\psi}{dt} \quad \text{by } ③$$

using this value in ④

$$\underline{a} = \frac{dv}{dt}\hat{t} + v\hat{n}\frac{d\psi}{dt}$$

$$= \frac{dv}{dt}\hat{t} + v\frac{d\psi}{ds}\cdot\frac{ds}{dt}\hat{n} \quad \text{(chain rule)}$$

$$= \frac{dv}{dt}\hat{t} + v\cdot v\frac{d\psi}{ds}\hat{n} \quad \because \frac{ds}{dt} = v$$

$$= \frac{dv}{dt}\hat{t} + v^2\cdot\frac{1}{\rho}\hat{n} \quad \because \frac{d\psi}{ds} = k = \frac{1}{\rho}$$

$\Rightarrow$ tangential Component of acc $= a_t = \dfrac{dv}{dt}$

Normal Component of acc $= a_n = \dfrac{v^2}{\rho}$

Remarks # ★ For different positions of particle during its motion the axes of $\hat{t}$ & $\hat{n}$ rotate in the form of a rigid frame about z-axis and its angular velocity is given by

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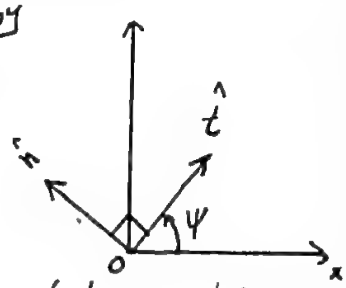
$$\underline{\omega} = \frac{d\psi}{dt}\hat{k}$$



★ The tangential Component of acc may also be expressed as

$$a_t = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} \quad \text{(Chain rule)}$$

$$a_t = v\frac{dv}{ds}$$

★ It is easily seen that the normal Component of acc always point to the Concave side of the

of the path:

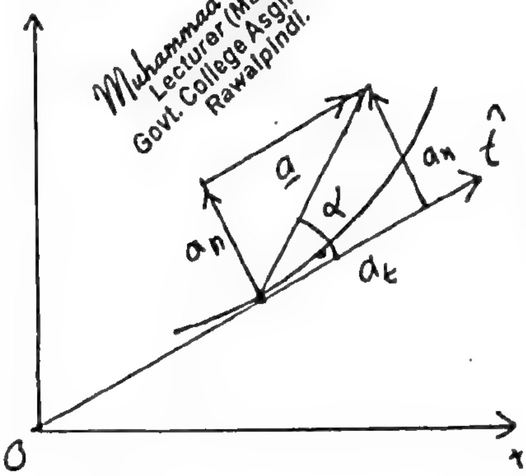
Magnitude and Direction of Acceleration in Intrinsic Co-ordinates

$$\underline{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

$$|\underline{a}| = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$$

Suppose acceleration vector $\underline{a}$ makes angle α with the tangential direction vector at P at any time t, then

$$\tan\alpha = \frac{v^2/\rho}{\left(\frac{dv}{dt}\right)}$$



Motion of a Particle in a Circle with Constant Speed

Consider a particle moving in a circle of radius r and with constant speed v.

$\because$ $v =$ constant

$\therefore$ tangential component of acceleration

$= a_t = \frac{dv}{dt} = 0$

i.e there is no acceleration along tangential direction.

Radius of curvature of circle $=$ radius $= r = \rho$

Normal Component of acc $= a_n = \frac{v^2}{\rho} = \frac{v^2}{r}$

Since perpendicular to tangent at circle passes through centre and normal acc is directed towards concave side of path, therefore acc $\frac{v^2}{r}$ is directed towards centre of circle.

Motion along straight line with Constant Speed

(Tangential and normal components of acceleration)

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Note that a straight itself is a tangent at each of its point.

Suppose a particle is moving along a straight line with a constant speed v. Then

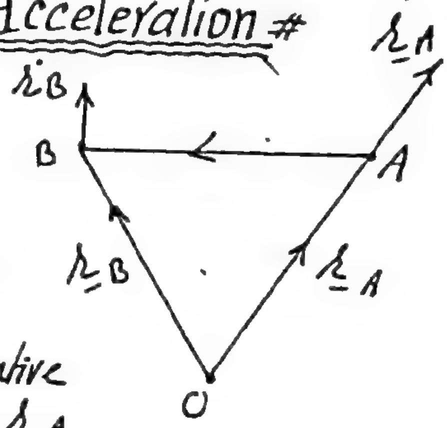
Tangential Component of acc $= \frac{dv}{dt} = 0 \quad (\because v = \text{Const})$

Radius of Curvature $= \infty$ because curvature is $k = 0$ & $\rho = \frac{1}{k} =$

Normal Component of acc $= a_n = \frac{v^2}{\rho} = \frac{v^2}{\infty} = 0$

Thus particle has no acceleration.

Relative Velocity and Acceleration



Let $\underline{r}_A$ and $\underline{r}_B$ be the position vectors of two particles A and B moving in xy-plane

Displacement of particle B relative to A $= \overrightarrow{AB} = \underline{r}_B - \underline{r}_A$

Relative velocity of B w.r.t A is defined by

$$\frac{d}{dt}(\overrightarrow{AB}) = \frac{d}{dt}(\underline{r}_B - \underline{r}_A)$$

$$= \frac{d\underline{r}_B}{dt} - \frac{d\underline{r}_A}{dt}$$

$$= \dot{\underline{r}}_B - \dot{\underline{r}}_A = \frac{d\underline{r}_{AB}}{dt}$$

$$= \underline{v}_B - \underline{v}_A$$

$$= (\text{Velocity of } B) - (\text{Velocity of } A)$$

Thus relative velocity of particle B relative to particle A is the vclocity which B appears to have when viewed from B

Relative velocity does not depend upon the the position of the two objects. but it depends upon their relative displacement

In case $\overrightarrow{AB}$ is a constant vector, then relative velocity of B is zero i.e

$$\frac{d}{dt}(\overrightarrow{AB}) = \underline{0}$$

$$\Rightarrow \dot{\underline{r}}_B - \dot{\underline{r}}_A = \underline{0}$$

$$\Rightarrow \underline{r}_B = \underline{r}_A$$

i.e velocities of objects are same

Thus if velocityes of objects are same, then their is no relative velocity of either object.

Relative acceleration of B relative to A is

$$\text{Relative acc. of } B = \frac{d^2\overrightarrow{AB}}{dt^2} = \ddot{\underline{r}}_B - \ddot{\underline{r}}_A$$

Remarks# * If we subtract velocity $\dot{\underline{r}}_A$ of A from velocities of A & B, then velocity A becomes zero and velocity of B becomes $\dot{\underline{r}}_B - \dot{\underline{r}}_A$ which is relative velocity of B w.r.t A

★ The velocity of B relative to A, $\underline{V}_B - \underline{V}_A$, is not equal to the velocity of A relative to B, $V_A - \underline{V}_B$, because these two vectors have opposite directions. However, the speed of B relative to A, $|\underline{V}_B - V_A|$, is equal to the speed of A relative to B, $|V_A - \underline{V}_B|$, because the relative speeds are scalar quantities.

Motion in Circle with Constan Angular Velocity

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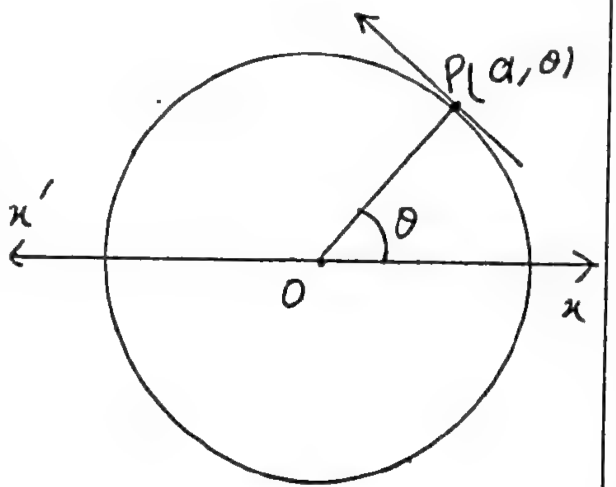
Problem # A particle is moving with constant angular velocity along a circle of radius 'a'. Find the radial and transverse components of acc of the particle and show that its acceleration is directed towards the centre of circle at each point of the circle.



Consider the motion of a particle moving in a circle of radius 'a'. Then equation of circle is

$$x^2 + y^2 = a^2$$

In polar form

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2$$

$$r^2 = a$$

$$r = a$$

Let the particle be at point $P(r, \theta) = (a, \theta)$ at any time 't'.

$\because \quad r = a$

$\Rightarrow \quad \dot{r} = 0 \quad \& \ \ddot{r} = 0$

Let ω be the constant angular velocity of the particle

$\because \quad \omega = \text{Constant}$

$\Rightarrow \quad \dot{\theta} = \text{Constant} \qquad \because \omega = \frac{d\theta}{dt}$

$\Rightarrow \quad \ddot{\theta} = 0$

Velocity in terms of radial and transverse components is

$$\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{s}$$

putting $\dot{r} = 0 \quad r = a, \ \dot{\theta} = \omega$

$$\underline{v} = 0 + a\omega\hat{s}$$

$$\underline{v} = a\omega\hat{s}$$

$\Rightarrow$ Linear velocity $= v = a\omega$ (in scalar form) and it is along the tanget at each point of the circle. Also since $a \ \& \ \omega$ are constant, therefore linear velocity is also constant.

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Acceleration#

acc. in radial and transverse components form is

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{s}$$

putting $\dot{r} = 0 = \ddot{r} \quad r = a \quad \dot{\theta} = \omega$

$$\underline{a} = (0 - a\omega^2)\hat{r} + (0 + 0)\hat{s}$$

$$= -a\omega^2\hat{r}$$

where $-ve$ sign shows that acceleration is always directed towards centre of the circle. Also there is no transverse acceleration of particle

Example # At any Time t, the position of a particle moving in a plane, can be specified by $(a\cos\omega t,\ a\sin\omega t)$, where a, ω are constants. Find the components of its velocity and acc along Co-ordinate axes.

Sol # At any time t the x-Co-ordinate of the position of particle is

$$x = a\cos\omega t$$

Hence the Component of velocity along x-axis

$$= v_x = \frac{dx}{dt} = -a\omega\sin\omega t$$

At any time t, the distance along y-axis is given by

$$y = a\sin\omega t$$

Hence the component of velocity along y-axis

$$v_y = \frac{dy}{dt} = a\omega\cos\omega t$$

Component of acc along x-axis is

$$a_x = \frac{d^2x}{dt^2} = -a\omega^2\cos\omega t$$

Component of acc along y-axis is

$$a_y = \frac{d^2y}{dt^2} = -a\omega^2\sin\omega t.$$

Example # A particle is moving along the parabola $x^2 = 4ay$ with constant speed v. Determine the tangential and the normal components of its acc when it reaches the point whose abscissa is $\sqrt{5}\,a$

Sol # The equation of the path of the particle is

$$x^2 = 4ay \longrightarrow ①$$

$\because$ $v =$ Constant (given)

$\therefore$ Tangential Component of acc $= a_t = \frac{dv}{dt} = 0$

Normal Component of acc

$$= a_n = \left.\frac{v^2}{\rho}\right|_{x=\sqrt{5}a} \longrightarrow ②$$

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Calculation of ρ at $x = \sqrt{5}a$

Formula for ρ is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} \longrightarrow ③$$

Differentiating ① w.r.t x

$$2x = 4a\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2a}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2a}$$

putting values of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in ③

$$\rho = \frac{\left[1+\left(\frac{x}{2a}\right)^2\right]^{3/2}}{\left|\frac{1}{2a}\right|} = \frac{\left[\frac{4a^2+x^2}{(2a)^2}\right]^{3/2}}{\frac{1}{2a}}$$

$$= \frac{[4a^2+x^2]^{3/2}}{(2a)^3} \times 2a$$

$$= \frac{[4a^2+x^2]^{3/2}}{(2a)^2}$$

Value of ρ at $x=\sqrt{5}a$

$$\rho\Big|_{x=\sqrt{5}a} = \frac{[4a^2+(\sqrt{5}a)^2]^{3/2}}{(2a)^2}$$

$$= \frac{(4a^2+5a^2)^{3/2}}{(2a)^2} = \frac{(9a^2)^{3/2}}{(2a)^2}$$

$$= \frac{[(3a)^2]^{3/2}}{(2a)^2} = \frac{(3a)^3}{4a^2} = \frac{27a^3}{4a^2}$$

$$= \frac{27a}{4}$$

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putting value of ρ in (2)

Normal Component of acc at $x=\sqrt{5}a$ $= \dfrac{v^2}{\frac{27a}{4}}$

$$= \frac{4v^2}{27a}$$

<u>Example</u> # A particle P moves in a plane in such a way that at any time t, its distance from a fixed point is $r = at+bt^2$ and the line connecting O and P makes an angle $\theta = ct^{3/2}$ with a fixed line OA. Find the radial and transverse components of the vel and acceleration of the particle at $t=1$

P.T.O

Sol # We have

$$r = at + bt^2 \qquad \theta = ct^{3/2}$$

$$\dot{r} = \frac{dr}{dt} = a + 2bt \qquad \dot{\theta} = \tfrac{3}{2}ct^{1/2}$$

$$\ddot{r} = \frac{d^2r}{dt^2} = 2b \qquad \ddot{\theta} = \tfrac{3}{4}ct^{-1/2} = \frac{3c}{4\sqrt{t}}$$

Radial Component of Velocity at any time $t = v_r = \dot{r}$

$$v_r = a + 2bt$$

putting $t = 1$

Radial Component of velocity at time $t = 1 = v_r = a + 2b$

Transverse Component of velocity at time $t = v_\theta = r\dot{\theta}$

$$v_\theta = (at + bt^2)\cdot\tfrac{3}{2}ct^{1/2}$$

putting $t = 1$

$$v_\theta\big|_{t=1} = (a+b)\,\tfrac{3}{2}c = \tfrac{3}{2}c(a+b)$$

Acceleration #

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Radial Component of acc at any time t is

$$a_r = \ddot{r} - r(\dot{\theta})^2$$

$$= 2b - (at + 2bt^2)\cdot\left(\tfrac{3}{2}ct^{1/2}\right)^2$$

putting $t = 1$

$$a_r\big|_{t=1} = 2b - (a + 2b)\cdot\left(\tfrac{3}{2}c\right)^2$$

$$= 2b - \frac{9c^2}{4}(a + 2b)$$

$$= \frac{1}{4}\left[8b - 9c^2(a+b)\right]$$

Transverse Component of acc at any time t is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$= (at+bt^2)\frac{3c}{4\sqrt{t}} + 2\cdot(a+2bt)\cdot\frac{3}{2}ct^{1/2}$$

putting $t=1$

$$a_\theta\big|_{t=1} = (a+b)\frac{3c}{4} + 2(a+2b)\frac{3}{2}c$$

$$= \frac{3}{4}c\left[a+b+4a+8b\right]$$

$$= \frac{3}{4}c(5a+9b)$$

Exercise

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Q: 2 # The position of a particle moving along an ellipse is given by

$$\underline{r} = a\cos t\,\hat{i} + b\sin t\,\hat{j}$$

If $a > b$, find the position of the particle where the velocity has a maximum or a minimum magnitude

Sol #

$$\underline{r} = a\cos t\,\hat{i} + b\sin t\,\hat{j} \longrightarrow (1)$$

$$\underline{V} = \frac{d\underline{r}}{dt} = -a\sin t\,\hat{i} + b\cos t\,\hat{j}$$

$$|\underline{V}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$= \sqrt{a^2\sin^2 t + b^2(1-\sin^2 t)}$$

$$|\underline{v}| = \sqrt{(a^2-b^2)\sin^2 t + b^2}$$

$\because \quad a > b \qquad \therefore \; a^2 > b^2$

$$\Rightarrow a^2 - b^2 > 0$$

Magnitude of velocity depends upon the factor $\sin^2 t$ and is maximum when $\sin^2 t$ is maximum i.e

if $\quad \sin^2 t = 1$

if $\quad \sin t = \pm 1$

if $\quad t = \pm\frac{\pi}{2}$

Putting $t = \pm\frac{\pi}{2}$ in ①, the position of the particle for maximum velocity is

$$\underline{r}\Big|_{t=\pm\frac{\pi}{2}} = a\cos\left(\pm\tfrac{\pi}{2}\right)\hat{i} + b\sin\left(\pm\tfrac{\pi}{2}\right)\hat{j}$$

$$= 0\hat{i} \pm b\hat{j}$$

$$= \pm b\hat{j}$$

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Min. Velocity # The velocity will be min if $\sin^2 t$ is min i.e

if $\quad \sin^2 t = 0$

if $\quad \sin t = 0$

if $\quad t = 0, \pi$

Putting these values of t in ①, the position of particle for min velocity is

$$\underline{r}\Big|_{t=0 \text{ or } \pi} = a\cos(0 \text{ or } \pi)\hat{i} + a\sin(0 \text{ or } \pi)\hat{j}$$

$$= \pm a\hat{i} + 0\hat{j} = \pm a\hat{i}$$

Q: 3# A particle is moving with uniform speed v along the curve

$$x^2y = a\left(x^2+\frac{a^2}{\sqrt{5}}\right)$$

Show that its acceleration has the maximum value $\frac{10v^2}{9a}$

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Sol# Equation of the path of particle is

$$x^2y = ax^2+\frac{a^3}{\sqrt{5}}$$

$$y = a+\frac{a^3x^{-2}}{\sqrt{5}} \longrightarrow (1)$$

Differentiating (1) w.r.t x

$$\frac{dy}{dx} = -\frac{2a^3x^{-3}}{\sqrt{5}}$$

$$\frac{d^2y}{dx^2} = \frac{6a^3x^{-4}}{\sqrt{5}}$$

Radius of curvature at any point (x,y) of the curve

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\left(-\frac{2a^3x^{-3}}{\sqrt{5}}\right)^2\right]^{3/2}}{\left(\frac{6a^3x^{-4}}{\sqrt{5}}\right)}$$

$$= \left[1+\frac{4a^6x^{-6}}{5}\right]^{3/2} \Big/ \frac{6a^3x^{-4}}{\sqrt{5}}$$

$$\rho = \frac{\left[1+\frac{4a^6}{5x^6}\right]^{3/2}}{\frac{6a^3}{\sqrt{5}x^4}}$$

$$= \left(\frac{5x^6+4a^6}{5x^6}\right)^{3/2} \cdot \frac{\sqrt{5}x^4}{6a^3}$$

$$= \frac{(5x^6+4a^6)^{3/2}}{5^{3/2}\cdot x^9} \cdot \frac{\sqrt{5}x^4}{6a^3}$$

$$= \frac{1}{30a^3}(5x^6+4a^6)^{3/2}x^{-5} \longrightarrow (2)$$

Since speed v is constant, therefore

$$v = \text{Constant}$$

$\Rightarrow$ tangential component of acc is

$$a_t = \frac{dv}{dt} = 0$$

Hence acc of particle is

$$\underline{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$$

$$= 0\hat{t} + \frac{v^2}{\rho}\hat{n}$$

$$a = |\underline{a}| = \frac{v^2}{\rho} \longrightarrow (3)$$

Now a will be maximum when ρ is minimum. Hence

$$a_{max} = \frac{v^2}{\rho_{min}} \longrightarrow (4)$$

Now we find min value of ρ

Min value of ρ

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from (2)

$$\rho = \frac{1}{30a^3}(5x^6+4a^6)^{3/2}x^{-5}$$

Differentiating w.r.t x

$$\frac{d\rho}{dx} = \frac{1}{30a^3}\left[\frac{3}{2}(5x^6+4a^6)^{1/2}\cdot 30x^5\cdot x^{-5} - 5x^{-6}(5x^6+4a^6)^{3/2}\right]$$

$$= \frac{1}{30a^3}\left[\cancel{60}\,15(5x^6+4a^6)^{1/2} - \frac{5}{x^6}(5x^6+4a^6)^{3/2}\right]$$

$$= \frac{1}{30a^3}\cdot 5(5x^6+4a^6)^{1/2}\left[\cancel{1}9 - \frac{1}{x^6}(5x^6+4a^6)\right]$$

$$= \frac{1}{6a^3}(5x^6+4a^6)^{1/2}\left[\frac{9x^6 - 5x^6 - 4a^6}{x^6}\right]$$

$$= \frac{1}{6a^3}(5x^6+4a^6)\left(\frac{4x^6-4a^6}{x^6}\right)$$

$$= \frac{4}{6a^3}(5x^6+4a^6)^{1/2}\left(\frac{x^6-a^6}{x^6}\right)$$

$$= \frac{2}{3a^3}(5x^6+4a^6)^{1/2}\left(\frac{x^6-a^6}{x^6}\right) \rightarrow (5)$$

For maximum & minum value of ρ

$$\frac{d\rho}{dx} = 0$$

$\Rightarrow$

$$\frac{2}{3a^3}\left(\frac{5x^6+4a^6)^{1/2}(x^6-a^6}{x^6}\right)=0$$

$$\Rightarrow \quad x^6-a^6=0 \quad \text{or} \quad 5x^6+4a^6=0$$

$$\Rightarrow \quad x=\pm a \quad \text{or} \quad x=\left(-\frac{4a^6}{5}\right)^{1/6}$$

which is imaginary and is not required

Thus $x=\pm a$

from ⑤

$$\frac{d\rho}{dx}=\frac{2}{3a^3}(5x^6+4a^6)^{1/2}\left(\frac{x^6-a^6}{x^6}\right)$$

$$=\frac{2}{3a^3}(5x^6+4a^6)^{1/2}(1-a^6x^{-6})$$

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Differentiating again w.r.t x

$$\frac{d^2\rho}{dx^2}=\frac{2}{3a^3}\Big[\frac{1}{2}(5x^6+4a^6)^{-1/2}\cdot 30x^5(1-a^6x^{-6})$$
$$+(5x^6+4a^6)^{1/2}(6a^6x^{-7})\Big]$$

$$\left.\frac{d^2\rho}{dx^2}\right|_{x=a}=\frac{2}{3a^2}\left[0+(5a^6+4a^6)^{1/2}(6a^6a^{-7})\right]>0$$

$$\& \left.\frac{d^2\rho}{dx^2}\right|_{x=-a}<0$$

$\Rightarrow \rho$ has min value at $x=a$

putting $x=a$ in ② min value of ρ is

$$\rho_{min}=\frac{(5a^6+4a^6)^{3/2}\cdot a^{-5}}{30a^3}=\frac{(9a^6)^{3/2}}{30a^3}\cdot a^{-5}$$

$$=\frac{27a}{30}=\frac{9a}{10}$$

putting this value of ρ_{min} in ④, we have maximum of acc as

$$a_{max} = \frac{v^2}{\frac{9a}{10}} = \frac{10v^2}{9a}$$

As required.

Q:4 # Find the tangential and normal components of acceleration of a point describing the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with uniform speed v when the particle is at $(0, b)$

$a_t = 0 \quad a_n = \frac{v^2 b}{a}$

Sol # The path of the particle is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow ①$$

$\because$ v is uniform

$\therefore$ $v =$ Constant

$\Rightarrow$ tangential component of acc $= a_t = \frac{dv}{dt} = 0$

Normal component of acc $= a_n = \frac{v^2}{\rho} \longrightarrow ②$

Value of ρ at $(0, b)$

Differentiating ① w.r.t x

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

at point $(0, b)$

$$\left(\frac{dy}{dx}\right)_{(0,b)} = -\frac{b^2}{a^2}\cdot\frac{0}{b} = 0$$

Again differentiating

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right]$$

at point $(0, b)$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,b)} = -\frac{b^2}{a^2}\left[\frac{b-0}{b^2}\right] = -\frac{b}{a^2}$$

Value of ρ at point $(0, b)$

$$\rho\Big|_{(0,b)} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2_{(0,b)}\right]^{3/2}}{\left|\left(\frac{d^2y}{dx^2}\right)_{(0,b)}\right|}$$

$$= \frac{(1+0)^{3/2}}{\left|-b/a^2\right|} = \frac{a^2}{b}$$

putting value of ρ in ②, normal Component of acc at $(0, b)$ is given by

$$a_n = \frac{v^2}{\frac{a^2}{b}} = \frac{bv^2}{a}$$

Q:5# Find the radial and transverse Components of the velocity of a particle moving along the curve at any time t if the polar angle $\theta = ct^2$

Ans $v_r = = \frac{(a-b)ct\sin 2\theta}{(a\cos^2\theta + b\sin^2\theta)^{3/2}}$ $\quad v_\theta = \frac{2ct}{\sqrt{a\cos^2\theta + b\sin^2\theta}}$

Sol# The equation of the path is

$$ax^2+by^2 = 1 \longrightarrow (1)$$

Since we use polar Co-ordinates to the radial and transverse components of vel. and acc, therefore we first change the equation of the path into polar form by putting

$$x = r\cos\theta \qquad y = r\sin\theta$$

in (1)

(1) becomes

$$a r^2\cos^2\theta + b r^2\sin^2\theta = 1$$

$$r = \frac{1}{\sqrt{a\cos^2\theta + b\sin^2\theta}} \longrightarrow (2)$$

Also $\quad \theta = ct^2 \longrightarrow (3)$

Now $\quad r = (a\cos^2\theta + b\sin^2\theta)^{-1/2}$

By chain rule

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$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left(a\cos^2\theta+b\sin^2\theta\right)^{-1/2}\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{2}\left(a\cos^2\theta+b\sin^2\theta\right)^{-3/2}\left(-2a\sin\theta\cos\theta+2b\sin\theta\cos\theta\right)\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{2}\left(a\cos^2\theta+b\sin^2\theta\right)^{-3/2}\left(-2a\sin\theta\cos\theta+2b\sin\theta\cos\theta\right)\cdot\dot{\theta}$$

$$= -\frac{1}{2}\cdot\frac{1}{(a\cos^2\theta+b\sin^2\theta)^{3/2}}(b-a)\sin 2\theta\cdot\dot{\theta}$$

from (3) $\quad \frac{d\theta}{dt} = 2ct \Rightarrow \dot{\theta} = 2ct$

using this value

$$\dot{r} = \frac{dr}{dt} = -\frac{1}{2} \quad \frac{(b-a)\,2ct\,\sin 2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}$$

$$= \frac{(a-b)\,ct\,\sin 2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}$$

Radial Component of velocity $= V_r = \frac{dr}{dt}$

$$= \frac{(a-b)\,ct\,\sin 2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}$$

Transverse Component of velocity

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$$= V_\theta = r\dot{\theta}$$

$$= \frac{1}{\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}} \cdot 2ct \qquad (\because \dot{\theta} = 2ct)$$

$$= \frac{2ct}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \qquad \text{As required}$$

Q:6 # Find the radial and transverse components of the acc of a particle moving along the circle

$$x^2 + y^2 = a^2$$

with constant angular velocity c

Ans $a_r = -ac^2$, $a_\theta = 0$

Sol # The path of the particle is

$$x^2 + y^2 = a^2$$

putting $x = r\cos\theta \qquad y = r\sin\theta$